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Disentangling q-Exponentials: A General Approach

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We revisit the q-deformed counterpart of the Zassenhaus formula, expressing the Jackson q-exponential of the sum of two non-q-commuting operators as an (in general) infinite product of q-exponential operators involving repeated q-commutators of increasing order, $E_q(A+B) = E_{q^{\alpha_0}}(A)E_{q^{\alpha_1}}(B)\prod_{i=2}^{\infty} E_{q^{\alpha_i}}(C_i)$. By systematically transforming the q-exponentials into exponentials of series and using the conventional Baker–Campbell–Hausdorff formula, we prove that one can make any choice for the bases q^{α_i}, $i = 0, 1, 2, \ldots$, of the q-exponentials in the infinite product. An explicit calculation of the operators C_i in the successive factors, carried out up to sixth order, also shows that the simplest q-Zassenhaus formula is obtained for $\alpha_0 = \alpha_1 = 1$, and $\alpha_2 = 2$, and $\alpha_3 = 3$. This confirms and reinforces a result of Sridhar and Jagannathan, on the basis of fourth-order calculations.

KEY WORDS: quantum groups; quantum algebras; q-exponential; q-Zassenhaus formula.

1. INTRODUCTION

Disentangling the exponential of the sum of two noncommuting operators into an (in general) infinite product of exponential operators involving repeated commutators of increasing order is a problem that occurs in many fields of physics, such as statistical mechanics, many-body theories, quantum optics, and path-integration techniques (see, e.g., Brif, 1996; Hatano and Suzuki, 1991; Suzuki, 1977; Wilcox, 1967; Witschel, 1975; Zhao, 1991). In particular, such a procedure may be employed to provide some useful approximation methods.

The problem is solved by applying the Zassenhaus formula, which was derived by Magnus (1954) citing unpublished work by Zassenhaus. This formula is the dual of the Baker–Campbell–Hausdorff (BCH) formula (Baker, 1902, 1903, 1904a,b; Campbell, 1898; Hausdorff, 1906), expressing the product of two noncommuting exponential operators as a single exponential operator in which the exponent is, in general, an infinite series in terms of repeated commutators.

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Since their advent (Drinfeld, 1987; Faddeev *et al.*, 1988; Jimbo, 1985, 1986), quantum groups and quantum algebras have had an ever-increasing and broader range of applications in mathematics and physics (see, e.g., Chaichian and Demichev, 1996; Klimyk and Schmüdgen, 1997; Majid, 1995). In developing noncommutative aspects of q-analysis, there has been a growing interest in getting q-deformed counterparts of standard results of conventional analysis, such as the BCH and Zassenhaus formulas.

In this respect, the simplest results are obtained for the Jackson q-exponential $E_q(z)$ (Jackson, 1904), on the basis of the use of the Heine basic numbers of base q, $[n]_q \equiv (1-q^n)/(1-q)$ (Heine, 1847). This function is often referred to as the maths-type q-exponential to distinguish it from the phys-type q-exponential for which the symmetric q-numbers $[n]_q \equiv (q^n - q^{-n})/(q - q^{-1})$ are employed.

A q-analogue of the BCH formula for $E_q(z)$ was derived by Katriel and Solomon (1991). Later on, Katriel *et al.* (1996) proposed a q-analogue of the Zassenhaus formula, wherein the q-exponential of the sum of two non-q-commuting operators is expressed as an (in general) infinite product of q-exponential operators involving repeated q-commutators of increasing order. Recently, Sridhar and Jagannathan (2002) derived another form of the q-Zassenhaus formula wherein, unlike in the Katriel *et al.* formula, the bases of the q-exponential factors in the infinite product are not the same. In both works, the operators in the successive factors were determined up to fourth order in the two operators.

Such results raise two questions: Can one make any choice of bases for the q-exponential factors in the q-Zassenhaus formula and, if so, for which choice of bases does the formula take the simplest form? It is the purpose of this paper to answer both of these questions.

To be able to carry out the analysis in general terms without making any choice of bases from the very beginning, we shall adopt another procedure than those previously employed. It is based on the repeated use of the conventional BCH formula after expressing every q-exponential as a standard exponential of a series (Hardy and Littlewood, 1946).

This paper is organized as follows. The conventional BCH and Zassenhaus formulas are reviewed in Section 2. In Section 3, after recalling the definition and main properties of the Jackson q-exponential, we determine the most general form of the q-Zassenhaus formula. In Section 4, the explicit form of the operators in the successive factors is determined up to sixth order to make an appropriate choice for the bases of the q-exponentials. Finally, Section 5 contains the conclusion.

2. CONVENTIONAL BCH AND ZASSENHAUS FORMULAS

In the next two sections, we shall make repeated use of the conventional BCH formula for the product of the exponentials of two (in general) noncommuting

operators X, Y,

$$\exp(X)\,\exp(Y) = \exp\left(\sum_{i=1}^{\infty} Z_i\right), \tag{2.1}$$

where Z_i is a homogeneous polynomial of degree i in X, Y (therefore said to be of ith order) and

$$Z_1 = X + Y. \tag{2.2}$$

Since for commuting operators X and Y, Z_1 is the only nonvanishing term in the series on the right-hand side of Eq. (2.1), it is obvious that for noncommuting operators, the additional terms $Z_2, Z_3, \ldots$, all contain the commutator $[X, Y]$. Following, for instance, the method given by Wilcox (1967), one can easily find the explicit expression of the BCH formula up to sixth order:

$$\begin{aligned}
Z_2 &= \frac{1}{2}[X, Y] \\
Z_3 &= \frac{1}{12}\big([X, [X, Y]] - [Y, [X, Y]]\big) \\
Z_4 &= -\frac{1}{24}[X, [Y, [X, Y]]] \\
Z_5 &= -\frac{1}{720}[X, [X, [X, [X, Y]]]] \\
&\quad -\frac{1}{120}[X, [X, [Y, [X, Y]]]] + \frac{1}{360}[Y, [X, [X, [X, Y]]]] \\
&\quad -\frac{1}{360}[X, [Y, [Y, [X, Y]]]] + \frac{1}{120}[Y, [X, [Y, [X, Y]]]] \\
&\quad +\frac{1}{720}[Y, [Y, [Y, [X, Y]]]] \\
Z_6 &= \frac{1}{720}[X, [X, [X, [Y, [X, Y]]]]] - \frac{1}{360}[X, [Y, [X, [X, [X, Y]]]]] \\
&\quad +\frac{1}{480}[Y, [X, [X, [X, [X, Y]]]]] - \frac{1}{480}[X, [X, [Y, [Y, [X, Y]]]]] \\
&\quad +\frac{1}{160}[X, [Y, [X, [Y, [X, Y]]]]] - \frac{1}{480}[Y, [X, [X, [Y, [X, Y]]]]] \\
&\quad +\frac{1}{1440}[Y, [Y, [X, [X, [X, Y]]]]] + \frac{1}{288}[X, [Y, [Y, [Y, [X, Y]]]]] \\
&\quad -\frac{1}{180}[Y, [X, [Y, [Y, [X, Y]]]]] + \frac{1}{360}[Y, [Y, [X, [Y, [X, Y]]]]].
\end{aligned} \tag{2.3}$$

The Zassenhaus formula, which we shall generalize to q-exponentials, can be written as

$$\exp(A+B) = \exp(A)\,\exp(B)\prod_{i=2}^{\infty}\exp(C_i) \tag{2.4}$$

where C_i is a homogeneous polynomial of degree i in A, B (therefore said to be of ith order). All the C_i's contain the commutator $[B, A]$ and, using Wilcox method (Wilcox, 1967) again, they can be determined up to sixth order:

$$\begin{aligned}
C_2 &= \frac{1}{2}[B, A] \\
C_3 &= \frac{1}{3}[[B, A], B] + \frac{1}{6}[[B, A], A] \\
C_4 &= \frac{1}{8}([[[B, A], B], B] + [[[B, A], A], B]) + \frac{1}{24}[[[B, A], A], A] \\
C_5 &= \frac{1}{30}([[[[B, A], B], B], B] + [[[[B, A], A], A], B]) \\
&\quad + \frac{1}{20}([[[[B, A], A], B], B] + [[[B, A], A], [B, A]]) \\
&\quad + \frac{1}{10}[[[B, A], B], [B, A]] \\
&\quad + \frac{1}{120}[[[[B, A], A], A], A] \\
C_6 &= \frac{1}{144}([[[[[B, A], B], B], B], B] + [[[[[B, A], A], A], A], B]) \\
&\quad + \frac{1}{72}([[[[[B, A], A], B], B], B] + [[[[B, A], A], A], B], B] \\
&\quad + [[[[B, A], A], A], [B, A]]) + \frac{1}{24}([[[[B, A], B], B], [B, A]] \\
&\quad + [[[[B, A], A], B], [B, A]]) + \frac{1}{720}[[[[[B, A], A], A], A], A].
\end{aligned} \tag{2.5}$$

It should be noted that the infinite series and products in this section and the next ones should be understood as formal ones and that their region of convergence should be studied for any specific choice of operators.

3. GENERAL FORM OF THE q-ZASSENHAUS FORMULA

The Jackson q-exponential[2] is defined by (Jackson, 1904)

$$E_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} \tag{3.1}$$

where

$$[n]_q \equiv \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1} \tag{3.2}$$

and

$$[n]_q! \equiv \begin{cases} 1 & \text{if } n = 0 \\ [n]_q[n-1]_q \ldots [1]_q & \text{if } n = 1, 2, \ldots \end{cases} \tag{3.3}$$

It has a finite radius of convergence $[\infty]_q = (1-q)^{-1}$ if $0 < q < 1$, but converges for all finite z if $q > 1$ (Exton, 1983). It is the eigenfunction of the Jackson q-differential operator

$$D_q E_q(\alpha z) = \alpha E_q(\alpha z) \tag{3.4}$$

where

$$D_q f(z) \equiv \frac{f(z) - f(qz)}{(1-q)z} \tag{3.5}$$

and it goes over to the conventional exponential for $q \to 1$.

In the appropriate region of definition, the q-exponential can be expressed as the exponential of a series

$$E_q(z) = \exp\left(\sum_{k=1}^{\infty} c_k(q) z^k\right) \tag{3.6}$$

where

$$c_k(q) = \frac{(1-q)^{k-1}}{k[k]_q}, \qquad k = 1, 2, 3, \ldots. \tag{3.7}$$

Although this formula can be traced back to Hardy and Littlewood (1946) and, as quoted in their paper, may even have been known of other mathematicians before, its simplicity and usefulness do not seem to have been fully appreciated in the physical literature. For this reason, in the Appendix, we provide a proof of the formula and derive from it some other interesting properties of the q-exponential.

[2] Jackson has actually introduced two different kinds of q-exponentials, related to one another by inversion. The function considered in Eq. (3.1) is connected to one of them. It is referred to as the Jackson q-exponential in most works of modern physics.

After these preliminaries, we are now in a position to generalize the Zassenhaus formula (2.4) to the q-exponential (3.1) and to obtain the following result.

Proposition. *For any choice of α_i, $i = 0, 1, 2, \ldots$, such that $\alpha_i \in R$, there exists a representation of the q-exponential of the sum of two operators A, B as an (in general) infinite product of the form*

$$E_q(A+B) = E_{q^{\alpha_0}}(A) \prod_{i=1}^{\infty} E_{q^{\alpha_i}}(C_i), \tag{3.8}$$

where $C_1 = B$ and C_i, $i = 2, 3, \ldots$, are some homogeneous polynomials of degree i in A, B.

Proof: Let $\alpha_0, \alpha_1, \alpha_2, \ldots$, be any set of real numbers and define the operator $G^{(0)}$ by

$$\begin{aligned} G^{(0)} &\equiv [E_{q^{\alpha_0}}(A)]^{-1} E_q(A+B) \\ &= \exp\left(-\sum_{k=1}^{\infty} c_k(q^{\alpha_0}) A^k\right) \exp\left(\sum_{k=1}^{\infty} c_k(q)(A+B)^k\right), \end{aligned} \tag{3.9}$$

where in the second step we used Eq. (3.6). The conventional BCH formula (2.1) with $X = -\sum_{k=1}^{\infty} c_k(q^{\alpha_0}) A^k$ and $Y = \sum_{k=1}^{\infty} c_k(q)(A+B)^k$ allows one to rewrite $G^{(0)}$ as

$$G^{(0)} = \exp\left(\sum_{k=1}^{\infty} G_k^{(0)}\right), \tag{3.10}$$

where $G_k^{(0)}$, $k = 1, 2, \ldots$, are some homogeneous polynomials of degree k in A, B, and $G_1^{(0)} = B$ since $c_1(q) = c_1(q^{\alpha_0}) = 1$.

Let now $G^{(1)}$ be defined by

$$G^{(1)} \equiv [E_{q^{\alpha_1}}(C_1)]^{-1} G^{(0)}, \qquad C_1 \equiv G_1^{(0)} = B. \tag{3.11}$$

On applying Eqs. (3.6) and (2.1) again, $G^{(1)}$ can be rewritten as

$$G^{(1)} = \exp\left(\sum_{k=2}^{\infty} G_k^{(1)}\right), \tag{3.12}$$

where $G_k^{(1)}$, $k = 2, 3, \ldots$, are some homogeneous polynomials of degree k in A, B, and there is no first-order term due to the choice made for C_1. Equations (3.9) and (3.11) together lead to the relation

$$E_q(A+B) = E_{q^{\alpha_0}}(A) E_{q^{\alpha_1}}(C_1) G^{(1)}, \tag{3.13}$$

where $G^{(1)}$ is given in (3.12).

Let us then assume that for some $j \in R^+$, the relations

$$E_q(A+B) = E_{q^{\alpha_0}}(A)\left(\prod_{i=1}^{j} E_{q^{\alpha_i}}(C_i)\right) G^{(j)} \tag{3.14}$$

$$G^{(j)} = \exp\left(\sum_{k=j+1}^{\infty} G_k^{(j)}\right) \tag{3.15}$$

hold for some homogeneous polynomials C_i (resp. $G_k^{(j)}$) of degree i (resp. k) in A, B. On setting

$$G^{(j+1)} \equiv [E_{q^{\alpha_{j+1}}}(C_{j+1})]^{-1} G^{(j)}, \qquad C_{j+1} \equiv G_{j+1}^{(j)} \tag{3.16}$$

and using Eqs. (3.6) and (2.1), we get

$$G^{(j+1)} = \exp\left(\sum_{k=j+2}^{\infty} G_k^{(j+1)}\right), \tag{3.17}$$

where $G_k^{(j+1)}$ are some homogeneous polynomials of degree k in A, B. Hence Eqs. (3.14) and (3.15) are valid when j is replaced by $j+1$. Furthermore, as shown in (3.12) and (3.13), they hold for $j = 1$. This therefore completes their proof by induction over j. □

For $j \to \infty$, we finally get the representation (3.8) of $E_q(A+B)$ as a formal infinite product.

4. EXPLICIT FORM OF THE q-ZASSENHAUS FORMULA UP TO SIXTH ORDER

In this section, we will apply the method presented in the previous one to determine the explicit form of the first few operators C_i. We will then make a choice for the bases q^{α_i} to get the simplest formula.

To start with, since the q-deformed counterpart of the multiplicative property of the conventional exponential reads (Cigler, 1979; Fairlie and Wu, 1997; Schützenberger, 1953)

$$E_q(A)E_q(B) = E_q(A+B) \quad \text{if} \quad [B, A]_q \equiv BA - qAB = 0, \tag{4.1}$$

it is convenient to choose $\alpha_0 = \alpha_1 = 1$ in Eq. (3.8). In this way, all the operators C_i, $i = 2, 3, \ldots$, will contain the q-commutator $[B, A]_q$ and therefore no terms depending only on A or B.

For $j = 0, 1, \ldots, n-1$ successively, we then determine the polynomials $G_k^{(j)}$, $k = j+1, j+2, \ldots, n$, of Eq. (3.15) up to some maximal order n. This provides us with some explicit expressions for $C_i = G_i^{(i-1)}$, $i = 2, 3, \ldots, n$, in terms

of A, B, and of the coefficients $c_k(q)$, $c_k(q^{\alpha_2})$, ..., defined in Eq. (3.7). Taking the latter into account, the C_i's are finally reexpressed in terms of q-commutators.

Let us illustrate the procedure by giving detailed results for $n = 3$. We successively get

$$
\begin{aligned}
G_1^{(0)} &= B \\
G_2^{(0)} &= c_2(q)B^2 + \left[c_2(q) + \frac{1}{2}\right] BA + \left[c_2(q) - \frac{1}{2}\right] AB \\
G_3^{(0)} &= c_3(q)B^3 + \left[c_3(q) + \frac{1}{2}c_2(q) - \frac{1}{12}\right] B^2A + \left[c_3(q) + \frac{1}{6}\right] BAB \\
&\quad + \left[c_3(q) - \frac{1}{2}c_2(q) - \frac{1}{12}\right] AB^2 + \left[c_3(q) + c_2(q) + \frac{1}{6}\right] BA^2 \\
&\quad + \left[c_3(q) - \frac{1}{3}\right] ABA + \left[c_3(q) - c_2(q) + \frac{1}{6}\right] A^2B \\
G_2^{(1)} &= \left[c_2(q) + \frac{1}{2}\right] BA + \left[c_2(q) - \frac{1}{2}\right] AB \\
G_3^{(1)} &= \left[c_3(q) - \frac{1}{3}\right] B^2A + \left[c_3(q) + \frac{2}{3}\right] BAB + \left[c_3(q) - \frac{1}{3}\right] AB^2 \\
&\quad + \left[c_3(q) + c_2(q) + \frac{1}{6}\right] BA^2 + \left[c_3(q) - \frac{1}{3}\right] ABA \\
&\quad + \left[c_3(q) - c_2(q) + \frac{1}{6}\right] A^2B \\
G_3^{(2)} &= G_3^{(1)}
\end{aligned}
\tag{4.2}
$$

from which we deduce that

$$
C_2 = G_2^{(1)} = \frac{1}{[2]_q}(BA - qAB) = \frac{1}{[2]_q}[B, A]_q \tag{4.3}
$$

and

$$
\begin{aligned}
C_3 &= G_3^{(2)} \\
&= \frac{1}{[3]_q}[-qB^2A + (1+q^2)BAB - qAB^2] \\
&\quad + \frac{1}{[3]_q!}[BA^2 - q(1+q)ABA + q^3A^2B] \\
&= \frac{1}{[3]_q}[[B, A]_q, B]_q + \frac{1}{[3]_q!}[[B, A]_q, A]_{q^2}.
\end{aligned}
\tag{4.4}
$$

Such expressions, which do not depend on $\alpha_2, \alpha_3, \ldots$, coincide with those given by Katriel *et al.* (1996) and by Sridhar and Jagannathan (2002).

With the help of Mathematica, we have calculated in the same way the next terms up to sixth order. They can be written as

$$C_4 = \frac{1}{[2]_q[4]_q}([[[B, A]_q, B]_q, B]_{q^2} + [[[B, A]_q, A]_{q^2}, B]_q) + \frac{1}{[4]_q!}[[[B, A]_q, A]_{q^2}, A]_{q^3} + \frac{q^a}{[2]_q[4]_q[2]_{q^a}}[[B, A]_q, [B, A]_q]_{q^{2-a}} \quad (4.5)$$

$$\begin{aligned} C_5 = {} & \frac{1}{[3]_q![5]_q}([[[[B, A]_q, B_q], B]_{q^2}, B]_{q^3} + [[[[B, A]_q, A]_{q^2}, A]_{q^3}, B]_q) \\ & + \frac{1}{[2]_q^2[5]_q}([[[[B, A]_q, A]_{q^2}, B]_q, B]_{q^2} + [[[B, A]_q, A]_{q^2}, [B, A]_q]_{q^2}) \\ & + \frac{1}{[2]_q[5]_q}[[[B, A]_q, B]_q, [B, A]_q]_{q^2} \\ & + \frac{1}{[5]_q!}[[[[B, A]_q, A]_{q^2}, A]_{q^3}, A]_{q^4} \end{aligned} \quad (4.6)$$

$$\begin{aligned} C_6 = {} & \frac{1}{[4]_q![6]_q}([[[[[B, A]_q, B]_q, B]_{q^2}, B]_{q^3}, B]_{q^4} \\ & + [[[[B, A]_q, A]_{q^2}, A]_{q^3}, A]_{q^4}, B]_q) \\ & + \frac{1}{[2]_q^2[3]_q[6]_q}([[[[[B, A]_q, A]_{q^2}, B]_q, B]_{q^2}, B]_{q^3} \\ & + [[[[B, A]_q, A]_{q^2}, A]_{q^3}, B]_q, B]_{q^2} \\ & + [[[[B, A]_q, A]_{q^2}, A]_{q^3}, [B, A]_q]_{q^2}) \\ & + \frac{1}{[2]_q^2[6]_q}([[[[B, A]_q, B]_q, B]_{q^2}, [B, A]_q]_{q^2} \\ & + [[[[B, A]_q, A]_{q^2}, B]_q, [B, A]_q]_{q^2}) \\ & + \frac{1}{[6]_q!}[[[[[B, A]_q, A]_{q^2}, A]_{q^3}, A]_{q^4}, A]_{q^5} \\ & + \frac{q^a}{[2]_q^2[6]_q[3]_{q^a}}[[[B, A]_q, [B, A]_q]_{q^{2-a}}, [B, A]_q]_{q^{2+a}} \\ & + \frac{q^b}{[3]_q[6]_q[2]_{q^b}}[[[B, A]_q, B]_q, [[B, A]_q, B]_q]_{q^{3-b}} \end{aligned}$$

$$+\frac{q^b}{[3]_q![6]_q[2]_{q^b}}([[[B,A]_q,B]_q,[[B,A]_q,A]_{q^2}]_{q^{3-b}}$$
$$+[[[B,A]_q,A]_{q^2},[[B,A]_q,B]_q]_{q^{3-b}})$$
$$+\frac{q^b}{[2]_q^2[3]_q[6]_q[2]_{q^b}}[[[B,A]_q,A]_{q^2},[[B,A]_q,A]_{q^2}]_{q^{3-b}}, \tag{4.7}$$

where we have set $a \equiv \alpha_2$, $b \equiv \alpha_3$. It can be easily checked that for $q \to 1$, Eqs. (4.3)–(4.7) give back the conventional results given in Eq. (2.5).

From such general results, it is clear that the simplest forms for C_4 and C_6 correspond to the choice $a = 2$, $b = 3$, in which case they become

$$C_4 = \frac{1}{[2]_q[4]_q}([[[B,A]_q,B]_q,B]_{q^2}+[[[B,A]_q,A]_{q^2},B]_q)$$
$$+\frac{1}{[4]_q!}[[[B,A]_q,A]_{q^2},A]_{q^3} \tag{4.8}$$

$$C_6 = \frac{1}{[4]_q![6]_q}([[[[[B,A]_q,B]_q,B]_{q^2},B]_{q^3},B]_{q^4}$$
$$+[[[[[B,A]_q,A]_{q^2},A]_{q^3},A]_{q^4},B]_q)$$
$$+\frac{1}{[2]_q^2[3]_q[6]_q}([[[[[B,A]_q,A]_{q^2},B]_q,B]_{q^2},B]_{q^3}$$
$$+[[[[[B,A]_q,A]_{q^2},A]_{q^3},B]_q,B]_{q^2}$$
$$+[[[[B,A]_q,A]_{q^2},A]_{q^3},[B,A]_q]_{q^2})$$
$$+\frac{1}{[2]_q^2[6]_q}([[[[B,A]_q,B]_q,B]_{q^2},[B,A]_q]_{q^2}$$
$$+[[[[B,A]_q,A]_{q^2},B]_q,[B,A]_q]_{q^2})$$
$$+\frac{1}{[6]_q!}[[[[[B,A]_q,A]_{q^2},A]_{q^3},A]_{q^4},A]_{q^5}, \tag{4.9}$$

while C_5 is independent of the choice made for the bases. Equation (4.8) agrees with Sridhar and Jagannathan (2002), who stopped at fourth order. The calculation of the next two terms, which we have carried out in this paper, strengthens the conjecture made by these authors according to which α_i, $i = 2, 3, \ldots$, should be taken as $\alpha_i = i$.

In contrast, it is clear from Eqs. (4.5) and (4.7) that the choice $\alpha_i = 1$, $i = 2, 3, \ldots$, made by Katriel *et al.* (1996) leads to much more complicated expressions

for C_4 and C_6, given by

$$C_4 = \frac{1}{[2]_q[4]_q}([[[B, A]_q, B]_q, B]_{q^2} + [[[B, A]_q, A]_{q^2}, B]_q) + \frac{1}{[4]_q!}[[[B, A]_q, A]_{q^2}, A]_{q^3} + \frac{q}{[2]_q^2[4]_q}[[B, A]_q, [B, A]_q]_q \quad (4.10)$$

$$\begin{aligned} C_6 = {} & \frac{1}{[4]_q![6]_q}([[[[[B, A]_q, B]_q, B]_{q^2}, B]_{q^3}, B]_{q^4} \\ & + [[[[[B, A]_q, A]_{q^2}, A]_{q^3}, A]_{q^4}, B]_q) \\ & + \frac{1}{[2]_q^2[3]_q[6]_q}([[[[[B, A]_q, A]_{q^2}, B]_q, B]_{q^2}, B]_{q^3} \\ & + [[[[[B, A]_q, A]_{q^2}, A]_{q^3}, B]_q, B]_{q^2} \\ & + [[[[B, A]_q, A]_{q^2}, A]_{q^3}, [B, A]_q]_{q^2}) \\ & + q[[[B, A]_q, [B, A]_q]_q, [B, A]_q]_{q^3} \\ & + q[[[B, A]_q, B]_q, [[B, A]_q, A]_{q^2}]_{q^2} \\ & + q[[[B, A]_q, A]_{q^2}, [[B, A]_q, B]_q]_{q^2}) \\ & + \frac{1}{[2]_q^2[6]_q}([[[[B, A]_q, B]_q, B]_{q^2}, [B, A]_q]_{q^2} \\ & + [[[[B, A]_q, A]_{q^2}, B]_q, [B, A]_q]_{q^2}) \\ & + \frac{1}{[6]_q!}[[[[[B, A]_q, A]_{q^2}, A]_{q^3}, A]_{q^4}, A]_{q^5} \\ & + \frac{q}{[3]_q![6]_q}[[[B, A]_q, B]_q, [[B, A]_q, B]_q]_{q^2} \\ & + \frac{q}{[2]_q^3[3]_q[6]_q}[[[B, A]_q, A]_{q^2}, [[B, A]_q, A]_{q^2}]_{q^2}, \end{aligned} \quad (4.11)$$

respectively. Equation (4.10) actually corrects their final expression, where we have found a misprint in one of the terms.

5. CONCLUSION

In this paper, we have revisited the q-deformed counterpart of the Zassenhaus formula that arises on replacing the conventional exponential of the sum of two noncommuting operators A and B by the Jackson q-exponential of the sum of two non-q-commuting ones.

We have proved that for any choice q^{α_i}, $\alpha_i \in R$, $i = 0, 1, 2, \ldots$, of the bases of the q-exponentials, there exists a representation of $E_q(A + B)$ as an (in general) infinite product $E_{q^{\alpha_0}}(A)E_{q^{\alpha_1}}(B)\prod_{i=2}^{\infty} E_{q^{\alpha_i}}(C_i)$, where C_i are some operators of increasing order in A, B.

To reproduce the multiplicative property of the q-exponential for q-commutative operators, we have then selected $\alpha_0 = \alpha_1 = 1$. With this choice and leaving the remaining α_i's arbitrary, we have finally obtained the explicit form of the C_i's for $i = 2, 3, \ldots, 6$. This has unambigously shown that $\alpha_2 = 2$ and $\alpha_3 = 3$ lead to the simplest q-Zassenhaus formula up to sixth order. Our work therefore confirms and reinforces the Sridhar and Jagannathan (2002) conjecture, derived from a study of fourth-order terms, according to which the best choice should be $\alpha_i = i$, $i = 2, 3, \ldots$.

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APPENDIX

In this Appendix we demonstrate Eqs. (3.6) and (3.7), and then use them to obtain some interesting properties of the q-exponential.

Proving Eqs. (3.6) and (3.7) amounts to finding the Taylor expansion of the logarithm of the q-exponential

$$\ln E_q(z) = \sum_{k=1}^{\infty} c_k(q) z^k. \tag{A1}$$

In Pourahmadi (1984) (see also Sachkov, 1996), it has been shown that if the functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $h(z) = \ln f(z) = \sum_{k=1}^{\infty} c_k z^k$ where $a_0 = 1$, are analytic in some neighborhood of zero, then the Taylor coefficients of $h(z)$ satisfy the recursion relation

$$c_k = a_k - \frac{1}{k}\sum_{j=1}^{k-1} j a_{k-j} c_j \qquad k = 2, 3, \ldots \tag{A2}$$

with $c_1 = a_1$. Applying this result to the logarithm of the q-exponential leads to the relations

$$c_k(q) = \frac{1}{[k]_q!} - \frac{1}{k}\sum_{j=1}^{k-1} \frac{j}{[k-j]_q!} c_j(q), \quad k = 2, 3, \ldots \tag{A3}$$

$$c_1(q) = 1. \tag{A4}$$

It is straightforward to check that the solution of Eq. (A3), satisfying condition (A4), is provided by Eq. (3.7). Inserting such an expression in Eq. (A3) converts the latter into the relation

$$\sum_{j=1}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q (1-q)^{j-1}[j-1]_q! = k, \quad k = 2, 3, \ldots \tag{A5}$$

where

$$\begin{bmatrix} k \\ j \end{bmatrix}_q \equiv \frac{[k]_q!}{[j]_q![k-j]_q!} \tag{A6}$$

denotes a q-binomial coefficient (Exton, 1983). Equation (A5) can be easily proved by induction over k by using the recursion relation

$$\begin{bmatrix} k \\ j \end{bmatrix}_q = q^j \begin{bmatrix} k-1 \\ j \end{bmatrix}_q + \begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_q, \quad j = 1, 2, \ldots, k-1. \tag{A7}$$

We indeed obtain

$$\begin{aligned}
&\sum_{j=1}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q (1-q)^{j-1}[j-1]_q! \\
&= \sum_{j=1}^{k-1} \begin{bmatrix} k-1 \\ j \end{bmatrix}_q q^j (1-q)^{j-1}[j-1]_q! \\
&\quad + \sum_{j=0}^{k-1} \begin{bmatrix} k-1 \\ j \end{bmatrix}_q (1-q)^j [j]_q! + (1-q)^{k-1}[k-1]_q! \\
&= \sum_{j=1}^{k-2} \begin{bmatrix} k-1 \\ j \end{bmatrix}_q q^j (1-q)^{j-1}[j-1]_q! + 1 \\
&\quad + \sum_{j=1}^{k-1} \begin{bmatrix} k-1 \\ j \end{bmatrix}_q (1-q)^{j-1}(1-q^j)[j-1]_q! \\
&= \sum_{j=1}^{k-1} \begin{bmatrix} k-1 \\ j \end{bmatrix}_q (1-q)^{j-1}[j-1]_q! + 1 \\
&= (k-1) + 1,
\end{aligned} \tag{A8}$$

where in the last step use has been made of the induction hypothesis.

Equations (3.6) and (3.7) can be applied to derive the following properties of the q-exponential, already quoted in Ubriaco (1992) (where we have corrected some misprints),

$$E_q(z)E_{q^{-1}}(z) = 1 \tag{A9}$$

$$E_q(z)E_q(-z) = E_{q^2}\left(\frac{1-q}{1+q}z^2\right) \tag{A10}$$

$$E_q([n]_{qz}) = \prod_{m=0}^{n-1} E_{q^n}(q^m z), \quad n = 2, 3, \ldots \tag{A11}$$

as well as a generalization of Eq. (A10),

$$\prod_{m=0}^{n-1} E_q(e^{2\pi i m/nz}) = E_{q^n}\left(\frac{(1-q)^{n-1}}{[n]_q}z^n\right), \quad n = 2, 3, \ldots \tag{A12}$$

The proof of these relations is based upon the multiplicative property of the ordinary exponential, $\exp(x)\exp(y) = \exp(x+y)$, and on some elementary properties of the coefficients $c_k(q)$, defined in (3.7),

$$c_k(q^{-1}) = (-1)^{k-1}c_k(q) \tag{A13}$$

$$2c_{2k}(q) = \left(\frac{1-q}{1+q}\right)^k c_k(q^2) \tag{A14}$$

$$[n]_{q^k}c_k(q^n) = ([n]_q)^k c_k(q) \tag{A15}$$

$$nc_{nk}(q) = \left(\frac{(1-q)^{n-1}}{[n]_q}\right)c_k(q^n). \tag{A16}$$

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